Differential Linear Matrix Inequalities Optimization

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***Abstract*—This letter proposes a new method to solve convex programming problems with constraints expressed by differential linear matrix inequalities (DLMIs). Initially, feasible solutions of interest are characterized and a general numerical method, based on the well known outer linearization technique, is proposed and discussed from theoretical and numerical viewpoints. Feasible solutions are written as a truncated series of a given set of time valued continuous functions with symmetric matrix coefficients to be determined. The numerical method encompasses the piecewise linear solution usually adopted in the literature with lower computational burden. In the sequel, several sampled-data control design problems whose optimality conditions can be expressed in this mathematical framework are provided. They are solved in order to put in evidence the most important aspects of the proposed method as well as to evaluate and compare numerical efficiency and limitations. Moreover, it is shown that DLMIs are particularly well adapted to cope with this class of optimal control design problems.**

***Index Terms*—Differential linear matrix inequalities (DLMI), optimal control, sampled-data control.**

I. Introduction

Differential linear matrix inequality (DLMI) is a mathematical entity expressed in the following general form

$$\mathcal{L}\big(\dot{X}(t), X(t)\big) < 0 \tag{1}$$

for all $t \in [0, h)$ with $h > 0$ being a given scalar that defines the time interval of interest. In (1), $X(t) : [0, h) \to \mathbb{R}^{n\times n}$ is a symmetric matrix function and the symbol $\mathcal{L}(\cdot, \cdot)$ denotes a linear matrix-valued function. Observe that convex differential constraints can also be expressed as (1) from the calculation of appropriate Schur Complements, [5]. Moreover, our main interest is to determine a solution (if any) that satisfies the boundary condition $(X_0, X_h) \in \Omega$ where $X(0) = X_0$, $X(h) = X_h$ and Ω is a convex set. To ease the notation, a feasible matrix trajectory $X(t)$ that satisfies the DLMI (1) for all $t \in [0, h)$, under the boundary condition $(X_0, X_h) \in \Omega$, is denoted simply as $X \in \mathcal{X}$. Hence, our main concern is to find the optimal solution to the following convex programming problem

$$f^* = \min_{X\in\mathcal{X}} f(X_0, X_h) \tag{2}$$

The importance of solving this problem for the class of matrix functions $X(t)$ to be specified later is enormous mainly because several optimal control design problems can be expressed of the form (2). Indeed, this is certainly the case of reference [12] where the usefulness of DLMIs in the framework of robust control design has been successfully pointed out for the first time many years ago. More recently, the results of [6], and [7] among others, by the same author, brought to light once again the importance of DLMIs and consequently the necessity to give more attention to the development of numerical methods. The situation is even more critical whenever we have to handle a large number of coupled DLMIs as (1), which often occur in many situations. As for instance, the sampled-data optimal control design of Markov jump linear systems reported in [8], where a large number of coupled DLMIs must be solved. Since the number of DLMIs equals the number of modes of the Markov process, the development of efficient numerical methods to deal with DLMIs appears to be highly justified.

The importance to treat control design problems expressed by DLMIs is shown in several recent papers as a consequence of the efforts to solve time-varying systems and finite-time horizon control problems, as in [3] and [13], respectively. In addition, in [2], a sufficient condition involving DLMIs has been given in order to characterize finite-time stability of linear time-varying systems with jumps. Specially, in [13], an important contribution was made where a DLMI was solved by splitting the time interval of interest into equally spaced time instants which enabled the determination of approximate solutions to $\mathcal{H}_\infty$ control problems with parameter uncertainty, see also [9]. Recently, the same strategy has been discussed in [6].

This letter concerns the numerical solution of (2). A general iterative algorithm based on the classical outer linearization

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technique is proposed and its convergence is proven. It depends on a given set of time-valued continuous functions whose main impact on optimality will be illustrated by means of several examples. The proposed algorithm guarantees that the differential inequality (1) is strictly satisfied in the whole time interval $[0, h)$, and optimality of the final solution within a desired precision level is attained by properly increasing the number of matrix variables to cope with. More precisely, the main contribution of this letter is the proposition of a new interactive algorithm able to determine a near-optimal solution to the convex programming problem (2) by adopting the following parametrization

$$X(t) = \sum_{i \in \mathbb{N}_\phi} X_i \phi_i(t) \tag{3}$$

where $\phi_i(t) : [0, h) \to \mathbb{R}$ for all $i \in N_\phi = \{0, 1, \dots, n_\phi\}$, with $n_\phi \geq 1$, constitute a set of given time-valued continuous functions and $X_i \in \mathbb{R}^{n\times n},\ i \in \mathbb{N}_\phi$ are symmetric matrices to be determined. In this sense, the piecewise solution proposed in [1] and more recently in [8], and polynomial solutions of finite degree emerging from sum of squares optimization, see [6], can be viewed as particular cases of our method that follow from adequate choices of the functions $\{\phi_i(\cdot)\}_{i\in\mathbb{N}_\phi}$. Hence, this letter proposes a new and alternative method to cope with convex programming problems of the form (2). However, it is important to stress that for the special case of polynomial functions the results of [10] for polynomials on convex domains and [11] for semidefinite programming are relevant and deserve more attention and research effort for comparison as far as numerical efficiency is concerned. Here, our main purpose is to provide an algorithm able to handle the convex problem (2) by adopting the approximation (3) where the functions $\phi_i(\cdot)$ for all $i \in \mathbb{N}_\phi$ are not necessarily polynomials. Of course, the determination of the best set of functions (in some sense) is an open problem to be addressed in the future.

The notation used throughout is standard. For real vectors or matrices, $(')$ refers to their transpose. The symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of real, real nonnegative and natural numbers, respectively. For symmetric matrices, $(\bullet)$ denotes the symmetric block. For any real symmetric matrix or symmetric matrix function the notation $Z(\cdot) > 0$ ($Z(\cdot) \geq 0$) indicates that it is positive (semi)definite. For a symmetric matrix $\sigma_{max}(\cdot)$ indicates its maximum eigenvalue.

II. Problem Statement and Preliminaries

Our main purpose is to provide a solution of (1), if any, for a class of real matrix functions that are useful in the context of optimal control systems design. To this end, the following definition is relevant to characterize the solutions of interest.

Definition 1 (DLMI Solution): The symmetric matrix function $P(t) : [0, h) \to \mathbb{R}^{n\times n}$ is a solution of (1) if $P(t)$ is continuous and satisfies

$$\mathcal{L}\big(\dot{P}(t), P(t)\big) < 0 \tag{4}$$

almost everywhere in the time interval $[0, h)$.

This definition preserves continuity but allows solutions $X(t)$ that fail to be differentiable in some isolated points of the time domain $[0, h)$. Moreover such DLMI, whenever admits a solution, is not unique in general. Hence, we search for feasible solutions of the form (3) where the continuous scalar valued functions $\{\phi_i(\cdot)\}_{i\in\mathbb{N}_\phi}$ define a basis for continuous scalar valued functions in the time interval $t \in [0, h)$. This means that any DLMI solution $P(t)$ can be approximated by $X(t)$ given in (3), provided that $X_i,\ i \in \mathbb{N}_\phi$ with n_ϕ possibly large enough, are properly determined. Several different sets of functions exhibiting this property can be adopted, among them, the simpler ones are as follows:

- *Piecewise linear -* The time interval $[0, h)$ is split into n_ϕ subintervals of length $\eta = h/n_\phi$. The scalar valued continuous function defined for all $t \in [0, h)$
$$\phi(t) = \begin{cases} 1 - \frac{|t|}{\eta}, & |t| \leq \eta \\ 0, & |t| > \eta \end{cases} \tag{5}$$
is used to construct the set of piecewise continuous linear functions $\phi_i(t) = \phi(t - i\eta)$ defined in the whole time interval $[0, h)$, for all $i \in \mathbb{N}_\phi$. Summing up all contributions we obtain
$$X(t) = X_i + \left(\frac{X_{i+1} - X_i}{\eta}\right)(t - i\eta) \tag{6}$$
valid in the time segment $t \in [i\eta, (i + 1)\eta)$ for each $i = 0, 1, \dots, n_\phi - 1$. Continuity is preserved but differentiability, in general, does not hold at the border points of each time subinterval.
- *Taylor series -* In the time interval $[0, h)$ consider the set of polynomial normalized functions $\phi_i(t) = (t/h)^i$ for all $i \in \mathbb{N}_\phi$. Hence, (3) rewritten as
$$X(t) = \sum_{i\in\mathbb{N}_\phi} X_i\left(\frac{t}{h}\right)^i \tag{7}$$
is the Taylor's expansion of a continuously differentiable DLMI solution $P(t)$ at $t = 0$ with successive matrix coefficients $X_i(i!/h^i)$, for all $i \in \mathbb{N}_\phi$.
- *Fourier series -* In the time interval $[0, h)$, setting $T > h$, the equation (3) has the form
$$X(t) = \sum_{i\in\mathbb{N}_\phi} X_i \cos\left(\left(\frac{\pi i}{T}\right)\left(t + \left(\frac{T-h}{2}\right)\right)\right) \tag{8}$$
which has been obtained from the following mathematical manipulations. Let $P_T(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a periodic even function with period $2T$. Extracting $P(t)$ from $P(t) = P_T(t + (T - h)/2)$ for all $t \in [0, h)$, the development of $P_T(t)$ in Fourier series provides $X(t)$ given by (8). This solution avoids the evaluation of the Fourier series at the border points of the time interval under consideration. We always have adopted $T = 2h$.

It is important to remark that the first choice (5) reproduces the piecewise linear solutions considered in [1] and [8]. Moreover, as we have mentioned before, many different sets of functions can be adopted. For instance, a set that combine the first and second or the first and third are other possibilities. The determination of the best set of functions that furnishes a near optimal solution to (2) with the smallest computational burden (represented by n_ϕ) is still an open problem to be addressed in the future. Actually, denoting $X \in \mathcal{X}_\phi$ a symmetric matrix function $X(t)$ of the form (3) that satisfies the DLMI (1) for all $t \in [0, h)$ under the boundary condition $(X_0, X_h) \in \Omega$, we want to find the optimal solution to the convex programming problem

$$f^* \leq f_\phi^* = \min_{X \in \mathcal{X}_\phi} f(X_0, X_h) \tag{9}$$

For a given set of functions $\phi_i(\cdot),\ i \in \mathbb{N}_\phi$, the corresponding suboptimal solution is such that $f^* \leftarrow f_\phi^*$ whenever, putting aside isolated points, $P(t) \leftarrow X(t)$ in the time interval $t \in [0, h)$. Clearly, the three possibilities considered before, among others, exhibit this important and necessary property whenever n_ϕ is chosen (possibly) sufficiently large.

III. Outer Linearization

This section is devoted to the development of a global convergent algorithm able to solve the convex programming problem stated in the right hand side of (9). The set of scalar valuated functions $\phi_i(\cdot),\ i \in \mathbb{N}_\phi$, is supposed to be known. The main idea is to use the notion of outer linearization, see [4], in order to decompose the original problem in a simpler convex problem expressed by linear matrix inequalities (LMIs). To this end, let us introduce the set of $m \geq 2$ ordered and different time samples

$$\tau_m = \{t_r \in [0, h] : r = 1, 2, \ldots, m\} \tag{10}$$

where $t_1 = 0$ and $t_m = h$ and the real scalars

$$d_{ir} = \frac{d\phi_i}{dt}(t_r),\ \ c_{ir} = \phi_i(t_r) \tag{11}$$

for all $r = 1, 2, \ldots, m$ and all $i \in \mathbb{N}_\phi$. Associated to τ_m, it is convenient to define the convex set

$$\mathcal{X}_m = \left\{ X_0, \ldots, X_{n_\phi} : \mathcal{L}\left(\sum_{i \in \mathbb{N}_\phi} X_i d_{ir}, \sum_{i \in \mathbb{N}_\phi} X_i c_{ir} \right) < 0 \right\} \tag{12}$$

for all $r = 1, \ldots, m$. It is clear that it has been obtained by imposing the inequality constraint (4) for each time instant $t \in \tau_m$, so being a convex set expressed in terms of m LMIs. The next lemma states a useful property, that is, the set $\mathcal{X}_\phi$ is a subset of $\mathcal{X}_m$.

Lemma 1: For any $m \geq 2$ then $\mathcal{X}_\phi \subseteq \mathcal{X}_m$.

Proof: It follows immediately from the fact that $\mathcal{X}_\phi = \lim_{m \to \infty} \mathcal{X}_m$ whenever the samples in the interior of the time segment are such that $\tau_\infty = \lim_{m \to \infty} \tau_m = [0, h]$. Hence, since $\tau_m \subseteq [0, h]$ for any given $m \geq 2$, the claim follows. ■

The set $\mathcal{X}_m$ for $m \geq 2$ is a result of an outer linearization procedure applied to the convex set $\mathcal{X}_\phi$ in the sense that the constraints of $\mathcal{X}_m$ are expressed through linear matrix inequalities and the subset relation $\mathcal{X}_\phi \subseteq \mathcal{X}_m$ holds for all $m \geq 2$. Hence, it is natural to take advantage of this property by introducing the auxiliary convex programming problem

$$f_m^* = \min_{X_0, \ldots X_{n_\phi} \in \mathcal{X}_m} f(X_0, X_h) \tag{13}$$

which can be solved with no big difficulty since it can be expressed, after linearizing the objective function by calculating Schur Complements if necessary, as a problem in the framework of semidefinite programming that can be handled by any LMI solver. The following globally convergent algorithm based on outer linearization is proposed. Indeed, it will be proven hereafter that, whenever it exists and is bounded, the proposed algorithm always converges to the optimal solution of problem (9).

- *Step 1:* Let the set of functions $\phi_i(\cdot),\ i \in \mathbb{N}_\phi$ be given. Set $m = 2$ and $\tau_m = \{0, h\}$.
- *Step 2:* Solve problem (13) to determine f_m^* and the symmetric matrices $X_0, \ldots, X_{n_\phi}$.
- *Step 3:* For the matrix function (3), calculate by line search the optimal solution (θ_{m+1}, t_{m+1}) of

$$\theta_{m+1} = \max_{t \in [0,h]} \sigma_{max}\big(\mathcal{L}\big(\dot{X}(t), X(t)\big)\big) \tag{14}$$

If $\theta_{m+1} < 0$ **Stop**. Otherwise, update $\tau_{m+1} = \tau_m \bigcup \{t_{m+1}\}$, set $m \leftarrow m + 1$ and go back to **Step 2**.

This algorithm is surprisingly simple and can be numerically implemented in a very efficient way by using standard LMI techniques, see [5], to handle problem (13) in step 2 and a line search procedure to determine the parameters (θ_{m+1}, t_{m+1}) in step 3. More details about numerical implementation and performance evaluation are given in the next section. For the moment, the next theorem shows that this is a globally convergent algorithm able to determine the optimal solution of the problem under consideration.

Theorem 1: Assume that the set $\mathcal{X}_\phi$ is non empty and f_ϕ^* is finite. The previous algorithm generates a sequence such that

$$f_m^* \leq f_{m+1}^* \leq f_\phi^* \tag{15}$$

for all $m \geq 2$. Moreover, it converges to the global optimal solution of (9).

Proof: The inequalities in (15) follow from the result of Lemma 1 which states that $\mathcal{X}_\phi \subseteq \mathcal{X}_m$, for all $m \geq 2$. Moreover, it is also true that $\mathcal{X}_{m+1} \subset \mathcal{X}_m$ for all $m \geq 2$. This last property follows from the fact that $\tau_m \subset \tau_{m+1}$ and the optimal solution of problem solved in step 2 is unfeasible on the subsequent interaction, whenever the convergence condition in step 3 is not satisfied. As a consequence, the algorithm proceeds until a feasible solution is generated in step 3. The sequence $\{f_m^*\}_{m \geq 2}$ converges because it is nondecreasing and bounded. Moreover, it converges to the optimal solution since it is feasible and its objective function is such that $\lim_{m \to \infty} f_m^* \leq f_\phi^*$. ■

The rationale behind this algorithm stems from the fact that the test in step 3, namely, $\theta_{m+1} < 0$ is used to detect if the current solution is feasible, that is, $X \in \mathcal{X}_\phi$. If not, the time $t_{m+1} \in [0, h]$ corresponding to the largest constraint violation is included in the set τ_{m+1} which makes the current solution unfeasible and forces the next solution to be feasible as far as all points belonging to τ_{m+1} are concerned. In general, this has a positive impact on the reduction of the number of iterations needed for convergence. The numerical behavior of the proposed algorithm will be illustrated by means of several examples reported in the next section.

Remark 1: The determination of θ_{m+1} in step 3 must be done with care. In order to be sure that $\theta_{m+1} < 0$ implies $\sigma_{max}(\mathcal{L}(\dot{X}(t), X(t))) < 0$ for all $t \in [0, h]$, the line search used in problem (14) must be implemented with a small step size δt, typically satisfying $\delta t \ll h$. In the numerical experiments reported in the next section, we have successfully considered $\delta t/h = 0.001$. As usual, a post-optimization feasibility test is recommended.

Remark 2: The complexity of the algorithm at interaction $m \geq 2$ can be measured by the number of LMIs which is equal to $m+2$ and the number of scalar variables $(n_\phi+1)(n+1)n/2$ where n_ϕ is the number of time subintervals for the piecewise linear solution, the degree of the truncated Taylor series expansion or the number of terms, excluding the constant one, of the Fourier series. In the next section, the impact of this important aspect on numerical performance will be analyzed for the three approximations previously discussed.

For the sake of comparison we now observe that the piecewise liner function (6) is continuous and can be imposed as a feasible solution to the problem under consideration. Indeed, denote $\mathcal{X}_{pw}$ the set $\mathcal{X}_\phi$ for the special case the functions $\phi_i(t) = \phi(t - i\eta)$ for all $i \in \mathbb{N}_\phi$ are given by (5). The next lemma summarizes an important property of this class of functions in the context of this letter.

Lemma 2: Let $n_\phi \geq 1$ be given. The condition $X \in \mathcal{X}_{pw}$ holds if and only if for each $i = 0, 1, \ldots, n_\phi - 1$, the LMIs

$$\mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_i\right) < 0 \tag{16}$$

$$\mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_{i+1}\right) < 0 \tag{17}$$

together with the boundary conditions $(X_0, X_{n_\phi}) \in \Omega$, are feasible.

Proof: First of all notice that, by construction of the piecewise function, $X(0) = X_0$ and $X(h) = X_{n_\phi}$ meaning that the boundary conditions are satisfied. The proof follows from the simple observation that the piecewise solution (6) valid in the time interval $t \in [i\eta, (i+1)\eta)$ for $i = 0, 1, \ldots, n_\phi - 1$ can be rewritten as the convex combination $X(t) = (1 - \alpha_i(t))X_i + \alpha_i(t)X_{i+1}$ where

$$\alpha_i(t) = \frac{t - i\eta}{\eta} \in [0, 1] \tag{18}$$

Hence, multiplying inequality (16) by $1 - \alpha_i(t)$, inequality (17) by $\alpha_i(t)$, summing up the results and taking into account that $\mathcal{L}(\cdot, \cdot)$ is linear it is seen that $X \in \mathcal{X}_{pw}$ which proves the sufficiency. Necessity is immediate since the inequalities (16) and (17) are equal to $\mathcal{L}(\dot{X}(t), X(t)) < 0$ evaluated at $t = i\eta$ and $t = (i+1)\eta$, respectively. The proof is concluded. ■

Two aspects should be remarked. First, whenever the $2n_\phi$ LMIs indicated in Lemma 2 are feasible then the corresponding piecewise solution $X(t)$ solves the DLMI whatever the value of $n_\phi \geq 1$. Second, this kind of solution, proposed in [1] and more recently in [8], is directly calculated by any LMI solver without the need of an interactive process. For this reason, in our opinion, it establishes an adequate numerical efficiency measure to compare the three previously discussed approximations as presented in the next section.

IV. Examples and Numerical Implementation

In this section the previous algorithm is used to solve a series of optimal control problems of increasing difficulty. It consists on the minimization of the $\mathcal{H}_\infty$ norm from the exogenous input w to the controlled output z of a closed-loop linear system governed by a state feedback sampled-data control given by

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{19}$$

$$z(t) = Cx(t) + Du(t) \tag{20}$$

$$u(t) = Lx(t_k), \ \forall t \in [t_k, t_{k+1}) \tag{21}$$

where $t_0 = 0$ and $t_{k+1} - t_k = h$ are evenly spaced sampling instants for all $k \in \mathbb{N}$ with sampling period $h > 0$. The time-valued functions $x(t) : \mathbb{R}_+ \to \mathbb{R}^n$, $u(t) : \mathbb{R}_+ \to \mathbb{R}$, $w(t) : \mathbb{R}_+ \to \mathbb{R}$, and $z(t) : \mathbb{R}_+ \to \mathbb{R}^{n+1}$ are the state, the control, the exogenous input, and the controlled output, respectively. The sampled-data state feedback gain matrix $L \in \mathbb{R}^{1\times n}$ is the design variable to be determined.

Following [8], where numerical issues have not been developed, defining the augmented matrices of compatible dimensions

$$F = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \ G' = \begin{bmatrix} C' \\ D' \end{bmatrix}, \ J = \begin{bmatrix} E \\ 0 \end{bmatrix} \tag{22}$$

the problem to be solved can be expressed as (2) where the objective function is given by $f(X_0, X_h) = \mu$ and the set of feasible solutions $\mathcal{X}$ is defined by the DLMI

$$\mathcal{L}(\dot{X}, X) = \begin{bmatrix} -\dot{X} + XF' + FX & XG' & J \\ \bullet & -I & 0 \\ \bullet & \bullet & -\mu I \end{bmatrix} < 0 \tag{23}$$

subject to the boundary condition $(X_0, X_h) \in \Omega$ defined by two coupled LMIs. One involving the initial condition $X(0) = X_0$, of the form

$$\begin{bmatrix} V & \begin{bmatrix} V & Y' \end{bmatrix} \\ \bullet & X_0 \end{bmatrix} > 0 \tag{24}$$

and one involving the final condition $X(h) = X_h$, that is

$$\begin{bmatrix} X_h & X_h \begin{bmatrix} I \\ 0 \end{bmatrix} \\ \bullet & V \end{bmatrix} > 0 \tag{25}$$

both related to the DLMI (23). It is to be noticed the presence of the scalar variable $\mu > 0$, the symmetric matrix variable $V \in \mathbb{R}^{n\times n}$ and the matrix variable $Y \in \mathbb{R}^{1\times n}$ which provide the minimum $\mathcal{H}_\infty$ cost and the associated sampled-data state feedback gain matrix $L = YV^{-1}$. For the numerical experience to be presented in the sequel, we have considered the following open-loop unstable system data whose complexity depends on the dimension n, that is, $h = 1$ [s] and

$$A = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0'_{n-1} \end{bmatrix}, \ B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}, \ E = 1_n \tag{26}$$

$$C = \begin{bmatrix} Q_n \\ 0'_n \end{bmatrix}, \ D = \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \tag{27}$$

where I_n, 0_n and 1_n denote the identity $n \times n$ matrix, the null $n \times 1$ vector and the one $n \times 1$ vector, respectively. Several $\mathcal{H}_\infty$ state feedback sampled-data optimal control problems with $n_\phi \in \{1, 2, 4, 8, 16\}$ for the piecewise linear approximation and $n_\phi \in \{1, 2, 3, 4, 5\}$ for the Taylor and Fourier series approximations have been solved with different matrices $Q_n \in \mathbb{R}^{n\times n}$.

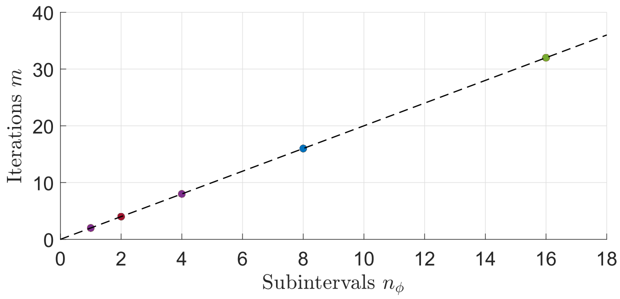

Fig. 1. Number of iterations for piecewise linear.

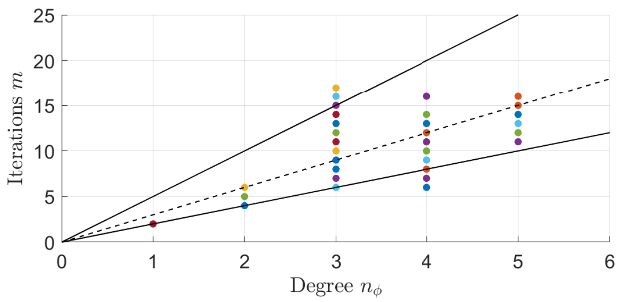

Fig. 2. Number of iterations for Taylor series.

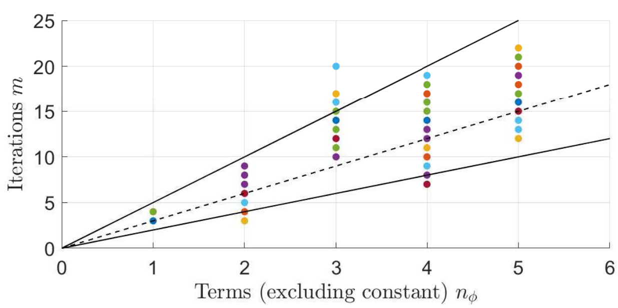

Fig. 3. Number of iterations for Fourier series.

The examples were solved using MATLAB LMI toolbox with default parameters in a computer with the following specifications: MacBook Pro (Mid 2012), 2.9 GHz Intel Core i7 processor and 8GB 1600 MHz DDR3 running the MATLAB R2016b version.

Figures 1, 2, and 3 show the number of iterations m needed for convergence of the proposed algorithm for piecewise linear, Taylor series and Fourier series approximations against n_ϕ, respectively. In Fig. 1, n_ϕ is the number of time subintervals, in Fig. 2 it is the degree of the polynomial and in Fig. 3 it is the number of terms excluding the constant one. In all figures, accordingly to (3), the quantity $n_\phi + 1$ equals the number of terms or which is the same, the number of free matrix variables of each approximation. We have considered $n \in \{2, 3, 4, 5\}$ and for each pair (n, n_ϕ) we have solved 20 different problems with Q_n being a square matrix with elements generated from a standard normal distribution. This gives the total number of 1,200 runs performed by the algorithm. Based on these figures, the following conclusions can be drawn:

- For the piecewise linear approximation the computational effort of the proposed algorithm and the solution provided by Lemma 2, solved in only one shot, are similar. Indeed, Fig. 1 shows that the algorithm converges for $m = 2n_\phi$ independently of n, the order of the open-loop system. This means that both procedures have to handle problems with the same number of variables and the same number of LMIs.
- Roughly speaking, for the Taylor series approximation, Fig. 2 shows that convergence is attained for $m \approx 3n_\phi$, independently of the order n considered. Notice the solid lines corresponding to $m = 2n_\phi$, $m = 5n_\phi$, and the dashed line corresponding to $m = 3n_\phi$. In general, the same numerical behavior is verified in Fig. 3 for the Fourier series approximation. Hence, at a first glance, as far as the computational effort is concerned this could lead to the conclusion that the three approximations are equivalent. However, this is not true as the next numerical experiments indicate.

TABLE I
MINIMUM $\mathcal{H}_\infty$ COST - (PIECEWISE LINEAR)

$n_\phi \backslash n$	2	3	4	5
1	527.35	2.82×10^4	∞	∞
2	122.96	2.70×10^3	8.53×10^4	∞
4	46.01	515.46	7.14×10^3	1.42×10^5
8	25.90	187.95	1.45×10^3	1.40×10^4
16	19.12	111.41	617.47	3.68×10^3

TABLE II
MINIMUM $\mathcal{H}_\infty$ COST - (TAYLOR SERIES)

$n_\phi \backslash n$	2	3	4	5
1	527.35	2.82×10^4	∞	∞
2	55.21	730.24	1.35×10^4	3.72×10^5
3	22.14	137.33	863.79	6.62×10^3
4	17.63	92.68	436.96	2.07×10^3
5	15.71	78.28	340.71	1.41×10^3

Tables I, II, and III give the value of the minimum $\mathcal{H}_\infty$ cost obtained with piecewise linear, Taylor series and Fourier series approximations for $Q_n = I_n$, respectively. As before, observe that $n \in \{2, 3, 4, 5\}$, and in all tables the parameter n_ϕ has the same meaning, that is, $n_\phi + 1$ is the number of free matrix variables of each approximation. Comparing the values in the three tables it is simple to notice that the Taylor and Fourier series approximations with $n_\phi = 4$ provide better results (in terms of the minimum $\mathcal{H}_\infty$ performance level) than the piecewise linear solution with $n_\phi = 16$. Since the computational burden depends on $m = 3n_\phi$ and $m = 2n_\phi$, respectively, the conclusion is that Taylor and Fourier series approximations are between two and three times better performing than the piecewise linear approximation calculated by the proposed algorithm or from the LMI conditions provided in Lemma 2. It can be viewed that the Taylor approximation performs a little bit better than the Fourier series approximation. It is clear that the dimension n determines the number of variables to be handled and, as a consequence, the computational burden. Moreover, in all tables, the symbol ∞ indicates that the corresponding minimum cost is very large so as the associated problem has been declared unfeasible by the algorithm, running with the stopping condition and convergence parameters as indicated before.

TABLE III
MINIMUM $\mathcal{H}_\infty$ COST - (FOURIER SERIES)

$n_\phi \backslash n$	2	3	4	5
1	1.24×10^3	1.09×10^5	∞	∞
2	101.69	1.92×10^3	5.61×10^4	∞
3	32.59	258.52	2.42×10^3	3.25×10^4
4	21.61	129.02	744.17	4.86×10^3
5	17.58	92.33	436.18	2.03×10^3

V. CONCLUSION

In this letter a new method for solving convex optimization problems subject to differential linear matrix inequalities with initial and final boundary conditions has been presented. It follows from a successful application of the classical outer linearization technique in a new framework, namely, the one defined by the specific class of optimization problems subject to DLMIs. It is shown that the proposed algorithm is globally convergent and requires to solve, in each iteration, a simple convex problem expressed through linear matrix inequalities. Numerical properties and comparisons are provided from the solution of several $\mathcal{H}_\infty$ optimal state feedback sampled-data control problems of increasing difficulty. The same kind of numerical analysis needs to be performed for the special case of polynomial basis whose feasibility and optimality may be alternatively tested from Handelman's theorem or sum of squares decomposition.

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